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LETTER TO THE EDITOR

Modulated structures of an Ising model with competing nearest-neighbour interactions

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Abstract. We use a two-dimensional mapping to analyse an Ising model with competing ferromagnetic and antiferromagnetic nearest-neighbour interactions on a Cayley tree. The phase diagram, which had some resemblance to the case of an Ising spin glass, displays some tricritical points and many modulated phases.

In order to explain the modulated structures of magnetic materials Yoshimori (1959), Villain (1959) and Kaplan (1959) proposed a spin model with nearest- and next-nearest-neighbour interactions. A simplified version, the axial next-nearest-neighbour Ising (ANNNI) model, was later introduced by Elliott (1961) and has received the attention of many authors (Bak and von Boehm 1980, Fisher and Selke 1980, Yokoi *et al* 1981) due to the fact that it exhibits a rich phase diagram with many modulated phases and a Lifshitz point. A similar model with nearest- and next-nearest-neighbour interactions has been analysed by Yokoi *et al* (1985) on a Cayley tree in the limit of infinite coordination number. It has been shown that it displays an analogous phase diagram with many modulated phases and also a Lifshitz point.

In this letter we analyse an Ising model on a regular Cayley tree of coordination z with nearest-neighbour interactions. The bonds are either ferromagnetic or antiferromagnetic in such a way that each spin interacts with n spins ferromagnetically and with $z-n$ spins antiferromagnetically. We have found that such a model, in spite of having only nearest-neighbour interactions, displays also a rich phase diagram with modulated phases and tricritical points. The version with coordination $z=3$ in the presence of a field was considered by Morita (1983) who argued that it could have the same properties of a certain model on a honeycomb lattice.

The model we analyse here has some resemblance to the spin glass model (Edwards and Anderson 1975, Sherrington and Kirkpatrick 1975) on a Cayley tree. In this system each interaction is either ferromagnetic or antiferromagnetic according to a certain prescribed probability and its properties are obtained by averaging over all configurations of bonds. Our model, however, corresponds to just one specified configuration of bonds. Although it includes frustration (the boundary spins are subject to an external field), it lacks the second ingredient necessary to obtain a spin glass phase, namely the configurational average. If we compare our results with the phase diagram of a spin glass we see that in the place of a spin glass phase we have a modulated phase.

Consider a regular Cayley tree with coordination number z with bonds of type 1 and type 2. Each site is connected to n sites by bonds of type 1 and to $z-n$ sites by bonds of type 2 (see figure 1). Following Morita (1983) we denote by $h_l^{(1)}$ ($h_l^{(2)}$) the effective field on a site which is connected to the innermost shell by a bond of type 1 (2). However, instead of writing the recursion relations in terms of these fields, we write them in terms of the variables $m_l^{(1)}$ and $m_l^{(2)}$ defined by

$$t_i m_l^{(i)} = \tanh \beta h_l^{(i)},$$

where $t_i = \tanh \beta J_i$, with J_1 and J_2 being the interactions associated with bonds of type 1 and 2, respectively. The recursion relations are then

$$m_{l+1}^{(1)} = \tanh\{(n-1)\tanh^{-1}(m_l^{(1)}t_1) + (z-n)\tanh^{-1}(m_l^{(2)}t_2)]$$
$$m_{l+1}^{(2)} = \tanh[n\tanh^{-1}(m_l^{(1)}t_1) + (z-n-1)\tanh^{-1}(m_l^{(2)}t_2)].$$

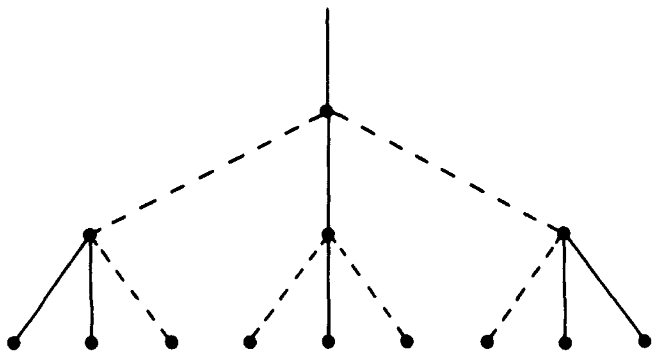

Figure 1. Branch of a Cayley tree with $z=4$ and $n=2$. The full and broken bonds denote interactions of type 1 and 2, respectively.

The paramagnetic phase corresponds to the region of the phase diagram where the trivial fixed point, $m^{(i)}=0$, of the mapping is stable. By a linear analysis we find that the boundary of this region is given by $\max_i|\lambda_i| = 1$, where λ_1 and λ_2 are the eigenvalues of the matrix M given by

$$M = \begin{bmatrix} (n-1)t_1 & (z-n)t_2 \\ nt_1 & (z-n-1)t_2 \end{bmatrix}.$$

Let A be the trace of M and B the determinant of M, that is,

$$A = (n-1)t_1 + (z-n-1)t_2$$

and

$$B = -(z-1)t_1 t_2.$$

If $A^2 \geqslant 4B$ then the critical line is given by $1-|A|+B=0$ and the paramagnetic phase borders a ferromagnetic phase if $A>0$, or an antiferromagnetic phase if $A<0$, since the eigenvalues are real. If $A^2<4B$ then the critical line is given by $B=1$ and the paramagnetic phase borders a modulated region since the eigenvalues are non-real.

Figure 2 shows the critical line for several values of z for a model defined by $J_1 = J > 0$ and $J_2 = -J$ (model 1). Figure 3 shows the critical line for some values of

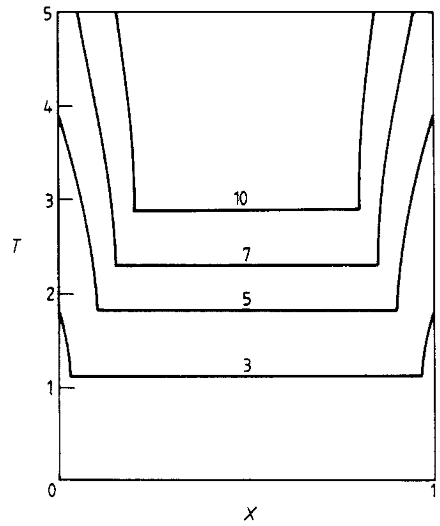

Figure 2. Critical line of model 1 for $z = 3, 5, 7$ and 10. The variable X is defined as $X = n/Z$.

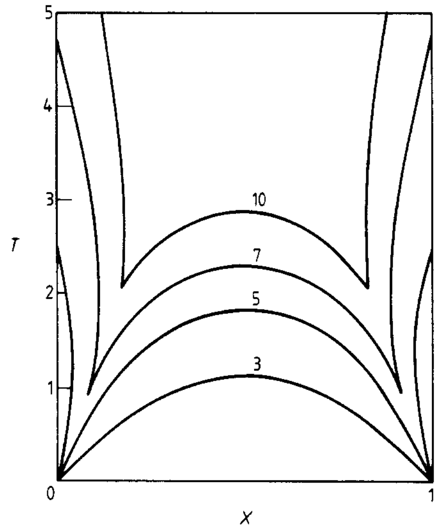

Figure 3. Critical line of model 2 for $z = 3$, 5, 7 and 10. The variable X is defined as $X = (1 + J_0/J)/2$.

z for a model defined by $J_1 = J_0 + J$ and $J_2 = J_0 - J$ with $J \geqslant J_0$ and $J > 0$, with $n = z/2$ (model 2). For the latter the ferromagnetic and antiferromagnetic phase appear only for $z > 4$.

We consider now the limit of infinite coordination number (Thompson 1982). For model 1 we take the limit $z \to \infty$, $n \to \infty$ and $J \to 0$ so that

$$(2n - z)/\sqrt{z} = p$$

and

$$\sqrt{z}J = \tilde{J}.$$

In this limit we get $A = p\beta\tilde{J}$ and $B = (\beta\tilde{J})^2$. For model 2 we let $z \to \infty$, $J_0 \to 0$ and $J \to 0$ in such a way that

$$zJ_0 = \tilde{J}_0$$

and

$$\sqrt{z}J = \tilde{J}.$$

In this case $A = \beta\tilde{J}_0$ and $B = (\beta\tilde{J})^2$. If in model 2 we define p by $p = \tilde{J}_0/\tilde{J}$, then both models give identical results in the limit considered. In particular, the critical line is given by

$$T^2 - |p|T + 1 = 0 \qquad \text{for } |p| \geq 2$$

and

$$T = 1 \qquad \text{for } |p| < 2$$

where $T = (\beta\tilde{J})^{-1}$.

Defining the variables

$$m_l = \tfrac{1}{2}(m_l^{(1)} + m_l^{(2)})$$

and

$$q_l = -(1/2\beta J)(m_l^{(1)} - m_l^{(2)})$$

the recursion relations are given by

$$m_{l+1} = \tanh((p/T)m_l - 1/T^2 q_l)$$

and

$$q_{l+1} = m_l[\operatorname{sech}((p/T)m_l - (1/T^2)q_l)]^2.$$

We have analysed in detail this two-dimensional mapping analytically as well as numerically and we have found results similar to those of Yokoi and de Oliveira (1985) and Yokoi *et al* (1985) for other models also on a Cayley tree. There are four types of attractors: (i) the trivial fixed point $m^* = 0$, $q^* = 0$ (which corresponds to the paramagnetic structure), (ii) a non-trivial fixed point $m^* \neq 0$ and $q^* \neq 0$ (ferromagnetic structure), (iii) periodic limit cycles (commensurate structure), and (iv) one-dimensional limit cycles (incommensurate structure). For a periodic limit cycle the principal wavenumber k is obtained as $2\pi n/N$ where n is the number of turns of the vector (m, q) in N iterations.

The global phase diagram (figure 4) displays a paramagnetic region, a ferromagnetic region, an antiferromagnetic region, and a modulated region. This last one is composed of an infinite number of smaller regions, characterised by a wavenumber $k/2\pi$, where the rational number k is in the interval [0, 1/2]. The transition from the paramagnetic to the modulated region results from a Hopf bifurcation and the discommensurations are due to tangent bifurcations (see Yokoi and de Oliveira 1985).

Figure 4 shows regions where more than one type of attractor exists which should be an indication of the presence of a first order transition. It shows also tricritical points and a Lifshitz point, actually an incipient one since another attractor is present which may correspond to a more stable phase. A more detailed analysis of the present model will be the subject of a forthcoming publication.

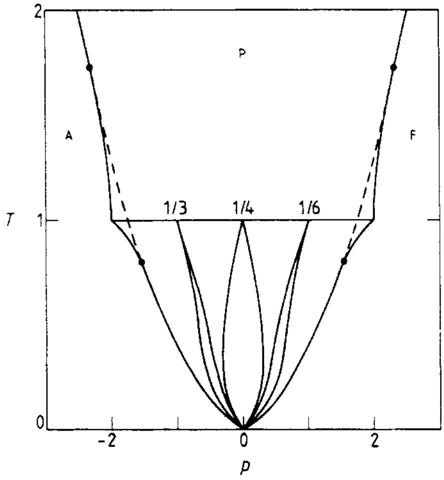

Figure 4. Global phase diagram in the limit of infinite coordination number. Paramagnetic (P), ferromagnetic (F), antiferromagnetic (A), and modulated (M) regions are shown. In the M region only a few commensurate phases are shown. The F and A regions extend up to the broken line overlapping the M and P regions. The four dots are tricritical points.

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